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On the Response of a Two-Level Quantum System to a Class of Time-Dependent Quasiperiodic Perturbations

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We study analytically the response of a two-level quantum system to a certain class of time-dependent quasiperiodic perturbations generated by a Fibonacci sequence. We show that the quasi-energy spectrum (Fourier transform of the evolution operator) generically is not a denumerable sum of delta functions. Hence the response is not quasiperiodic. Several numerical investigations (Poincaré sections, polarization fluctuation, etc.) suggest an intermediate kind of behavior between quasiperiodic and chaotic.

KEY WORDS: Fibonacci sequence; quasiperiodicity; two-level systems; quantum chaos.

1. INTRODUCTION

A bounded quantum system responds quasiperiodically to an external time-dependent periodic driving force. This behavior is due to the invariance of the total Hamiltonian under translation in time by the period of the driving force. The Floquet theorem[1] guarantees that every wave function $|\Phi(t)\rangle$ can be written as a linear combination of vectors of the form $e^{i\omega_\alpha t}\,|\phi_\alpha(t)\rangle$, where $|\phi_\alpha(t)\rangle$ are periodic, independent, and span the Hilbert space. For a bounded system, this implies that the frequency set $\{\omega_\alpha\}$ is denumerable, and hence $|\Phi(t)\rangle$ is quasiperiodic.

Once a quasiperiodic driving is applied, the system is no longer invariant under time translation, and a generalization of the Floquet

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theorem is not ensured to exist. Therefore, the problem is far more subtle, and the response of the system is not necessarily quasiperiodic, and may to some extent show chaotic features.

Recent experimental[2] and numerical[3–7] work has revived interest in the behavior of quantum systems driven by nonperiodic perturbations. Measurements of the ionization of high-n hydrogen atoms induced by bichromatic microwave fields have been recently reported.[2] The ionization data show typical quantum features, which can be interpreted to imply that in this problem the response is quasiperiodic.[7] Numerical studies try to characterize the response by analyzing the power-spectrum or various correlation functions of the solution. These numerical experiments provide some evidence for the existence of a transition from a quasiperiodic to a chaotic regime,[3,4] but the numerical work is fraught with nontrivial problems, and the extraction of unambiguous statements from the numerics requires great care.[5,6]

The purpose of the present paper is to report on an analytical treatment of a class of quasiperiodically driven two-level systems. We show that in this case the Fourier transform of the evolution operator is not a denumerable sum of delta functions. This analytical result is supplemented with some numerical evidence that the evolution exhibits some intermediate kind of behavior between quasiperiodic and chaotic. We believe that, in spite of the fact that we discuss a rather restricted and schematic model, the present results contribute toward an understanding of the response of a general quantum system to a quasiperiodic driving force.

We consider a two-level system with a Hamiltonian given by

$$H = E\sigma_z + g(t)\,\sigma_x \tag{1a}$$

with a driving perturbation of the form

$$g(t) = g_{\nu(k)}(t) \qquad \text{for} \quad t \in [k-1, k[\tag{1b}$$

where $\nu(k)$ can be either 0 or 1, and $g_{0,1}(t)$ are two arbitrary functions defined in the unit t interval. The function $g(t)$ can be made periodic (P), quasiperiodic (QP), or random (R) by choosing the sequence $\nu(k)$ as a P, QP, or R sequence of the symbols 0 and 1, respectively.

The Fourier transform of the evolution operator

$$G(t, 0) = T\exp\left\{-i\int_0^t H(t')\,dt'\right\}$$

is defined by

$$\tilde{G}(\omega) = \lim_{\substack{n\to\infty \\ \eta\to +0}} \int_0^{n+i\eta} e^{i\omega t} G(t, 0)\,dt \tag{2}$$

It takes the form

$$\tilde{G}(\omega) = \sum_{k=0}^{\infty} e^{i\omega k} V_{\nu(k+1)}(\omega)\, U_{\nu(k)} \cdots U_{\nu(1)} \tag{3}$$

where $U_{0,1}$ is the unitary evolution operator for the Hamiltonian H on a unit time interval, and

$$V_{0,1}(\omega) = \int_0^1 dt\, e^{i\omega t} T \exp\left\{ -i \int_0^t H(t')\, dt' \right\} \tag{4}$$

The nature of the solution (P, QP, or chaotic) is entirely determined by the nature of $\tilde{G}(\omega)$. For further reference, we define

$$W_k = U_{\nu(k)} U_{\nu(k-1)} \cdots U_{\nu(1)} \tag{5}$$

and

$$\tilde{G}_p(\omega) = \sum_{k=0}^{p} e^{i\omega k} V_{\nu(k+1)}(\omega)\, W_k \tag{6a}$$

so that

$$\tilde{G}(\omega) = \lim_{p \to \infty} \tilde{G}_p(\omega) \tag{6b}$$

Noting that, in the random case, the response is trivially chaotic, we shall discuss separately the P and QP driving forces.

2. PERIODIC DRIVING

If the sequence $\nu(k)$ is periodic with a period λ, it follows from (3) that

$$\tilde{G}(\omega) = \tilde{G}_\lambda(\omega) + e^{i\lambda\omega}[\tilde{G}(\omega) - V_{\nu(1)}(\omega)]\, W_\lambda \tag{7}$$

which can be solved to give

$$\tilde{G}(\omega) = V_{\nu(1)}(\omega) + [\tilde{G}_\lambda(\omega) - V_{\nu(1)}(\omega)](1 - e^{i\lambda\omega} W_\lambda)^{-1} \tag{8}$$

W_λ is a unitary matrix with eigenvalues $e^{\pm i\xi}$. Since $\tilde{G}_\lambda(\omega)$ and $V_{\nu(1)}(\omega)$ are regular by construction [see (4) and (6)], $\tilde{G}(\omega)$ has pole singularities at

$$\omega_l^{(\pm)} = \frac{1}{\lambda}(\pm\xi + 2\pi l) \qquad (l \text{ integer}) \tag{9}$$

This is the expression of the Floquet theorem in the present context. Note that, when the period λ increases, the support of $\tilde{G}(\omega)$ becomes denser, but it always consists of a denumerable set of isolated points.

3. QUASIPERIODIC DRIVING

We now discuss a particular QP sequence of functions $g_{\nu(k)}(t)$, namely the Fibonacci sequence, which is often used as a prototype of one-dimensional quasicrystals.[8–10] This sequence is constructed by the following substitution ("inflation rule"). Let F_L stand for the Lth Fibonacci number, defined by the recursion $F_L = F_{L-1} + F_{L-2}$ ($F_0 = 0$; $F_1 = 1$). We define a sequence of words M_L of lengths F_L, composed of the letters 0 and 1. Starting with $M_1 = \{0\}$ and $M_2 = \{1\}$, we construct the next words recursively by the concatenation rule

$$M_{L+1} = M_L M_{L-1} \tag{10}$$

The infinite sequence of indices $\nu(k)$ needed for the definition of the driving function $g(t)$ is given by the word M_L when $L \to \infty$. It is well known that this sequence is QP.[8–10] Let us prove it by an alternative method which will be useful in the following. This approach, due to Bombieri and Taylor,[11] has also been used recently in the study of some one-dimensional geometrical models.[12]

For the sake of simplicity we assume that the building blocks $g_{0,1}(t)$ have the form

$$g_{0,1}(t) = a_{0,1}\gamma(t) \tag{11}$$

where $a_{0,1}$ are constants, and $\gamma(t)$ is defined in the unit t interval.

The Fourier transform of $g(t)$ is given by

$$\tilde{g}(\omega) = \tilde{\gamma}(\omega) \sum_{k=1}^{\infty} a_{\nu(k)} e^{i\omega k} \tag{12a}$$

where

$$\tilde{\gamma}(\omega) = e^{-i\omega} \int_0^1 dt\, e^{i\omega t}\gamma(t) \tag{12b}$$

Consider

$$\tilde{g}_L(\omega) = \sum_{k=1}^{F_L} a_{\nu(k)} e^{i\omega k} \tag{13}$$

It satisfies the recurrence relation

$$\tilde{g}_{L+1}(\omega) = \tilde{g}_L(\omega) + e^{i\omega F_L}\tilde{g}_{L-1}(\omega) \tag{14}$$

due to the concatenation rule (10). Following the argument of Bombieri and Taylor,[11] we consider

$$f_L(\omega) = \tilde{g}_L(\omega)/F_L \tag{15}$$

which satisfies

$$f_{L+1}(\omega) = \left(1 + \frac{F_{L-1}}{F_L}\right)^{-1} \left[f_L(\omega) + \frac{F_{L-1}}{F_L} e^{i\omega F_L} f_{L-1}(\omega) \right] \tag{16a}$$

If, for a certain value of ω, the limit $f(\omega) = \lim_{L\to\infty} f_L(\omega)$ exists, then it must satisfy the equation

$$f(\omega) = f(\omega)(1 + \tau^{-1})^{-1} \, (1 + \tau^{-1} \lim_{L\to\infty} e^{i\omega F_L}) \tag{16b}$$

Here, $\tau = (\sqrt{5} + 1)/2 = \lim_{L\to\infty} F_{L+1}/F_L$ is the golden mean. Thus, all the points such that $f(\omega) \neq 0$ must satisfy

$$\lim_{L\to\infty} e^{i\omega F_L} = 1 \tag{17}$$

A standard result then gives $\omega = 2\pi(j + k\tau)$, where j and k are integers. This is a denumerable set of points, and therefore $\tilde{g}(\omega)$ is a denumerable sum of delta functions, which proves the QP nature of $g(t)$.

By a proper choice of the functions $g_{0,1}(t)$, one can make $g(t)$ to be infinitely differentiable on the real axis (it cannot be analytic because of the concatenation procedure). Thus, the amplitudes of the delta functions in $\tilde{g}(\omega)$ at $\omega = 2\pi(j + k\tau)$ (which are independent of j) can be made to decrease faster than any desired power of k.

We now return to the discussion of the nature of the Fourier transform of the evolution operator $\tilde{G}(\omega)$. We define

$$\varphi_L(\omega) = \tilde{G}_L(\omega) - V_{\nu(1)}(\omega) \tag{18}$$

and, in analogy with the derivation of Eq. (14), we have

$$\varphi_{L+1}(\omega) = \varphi_L(\omega) + e^{i\omega F_L} \varphi_{L-1}(\omega) \, W_{F_L} \tag{19}$$

Define now

$$\Phi_L(\omega) = \varphi_L(\omega)/F_L \tag{20a}$$

Then,

$$\Phi_{L+1}(\omega) = \left(1 + \frac{F_{L-1}}{F_L}\right)^{-1} \left[\Phi_L(\omega) + \frac{F_{L-1}}{F_L} \Phi_{L-1}(\omega) \, e^{i\omega F_L} W_{F_L} \right] \tag{20b}$$

Let

$$\Phi(\omega) = \lim_{L\to\infty} \Phi_L(\omega) \tag{20c}$$

Then

$$\Phi(\omega) = \Phi(\omega)(1+\tau^{-1})^{-1}\,(1+\tau^{-1}\lim_{L\to\infty} e^{i\omega F_L} W_{F_L}) \tag{20d}$$

As was argued above, $\Phi(\omega) \neq 0$ if, and only if, $\lim_{L\to\infty} e^{i\omega F_L} W_{F_L}$ has a (left) eigenvalue equal to 1. Since W_{F_L} is a unitary operator with unit determinant, its eigenvalues are of the form $e^{\pm i\xi_L}$, and the condition for $\Phi(\omega)$ to be nonzero is

$$\lim_{L\to\infty} (\xi_L \pm \omega F_L) = 0 \bmod 2\pi \tag{21a}$$

Defining $x_L = \frac{1}{2}\operatorname{Tr} W_{F_L} \equiv \cos \xi_L$, we get

$$\lim_{L\to\infty} (x_L - \cos \omega F_L) = 0 \tag{21b}$$

The traces of the Fibonacci concatenated matrices W_{F_L} are known to obey the recursion relation[13–15]

$$x_{L+2} + x_{L-1} = 2x_{L+1}x_L \tag{22a}$$

This relation has an invariant

$$I = x_{L+1}^2 + x_L^2 + x_{L-1}^2 - 2x_{L+1}x_Lx_{L-1} - 1 \tag{22b}$$

It can be easily seen that, if x_L is asymptotic to $\cos \omega F_L$, the invariant I must converge to zero. Hence, it must be strictly zero, for any L.

The two evolution operators U_0 and U_1 used for the construction of W_{F_L} can be represented as

$$U_0 = \exp(i\boldsymbol{\sigma}\cdot\mathbf{k}_0), \qquad U_1 = \exp(i\boldsymbol{\sigma}\cdot\mathbf{k}_1) \tag{23a}$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices and $\mathbf{k}_0$ and $\mathbf{k}_1$ are three-dimensional vectors making an angle θ.

In terms of k_0, k_1, and θ, the invariant I takes the form

$$I = -\sin^2\theta \sin^2 k_0 \sin^2 k_1 \tag{23b}$$

which vanishes only for trivial cases. Hence, x_L cannot be asymptotic to $\cos \omega F_L$, and the equality in (20d) can be satisfied only for $\Phi(\omega) = 0$. This proves that the Fourier transform (quasi-energy spectrum) $\tilde{G}(\omega)$ generically does not contain any delta function, and therefore $G(t)$ is not quasiperiodic.

For certain values of the parameters, the trace map (22a) has cycles and the evolution operator taken at the Fibonacci numbers forms a periodic sequence. However, the system is still chaotic, since the evolution operator taken at all integer times is neither periodic nor even quasiperiodic. Hence these cases are not exceptions to our general proof.

In analogy with previous work on electron and phonon spectra (see refs. 13–15), we may suggest that the spectrum of $\tilde{G}(\omega)$ is generically purely singular continuous. Evidence in favor of this statement comes from the following numerical studies. These have been performed for a special class of perturbation potentials $g(t)$, namely "kicks," i.e., delta functions at integer times. With the notation of Eq. (11), we have $\gamma(t) = \delta(t)$, and the coupling constants a_0, a_1 are chosen as follows:

$$a_0 = -\tau a; \qquad a_1 = a \tag{24}$$

in such a way that the sequence $a_{\nu(k)}$ has zero average. The invariant I of the trace map (22b) then reads

$$I = -\sin^2 E \sin^2(\tau^2 a) \tag{25}$$

It follows from the above analysis that the quasi-energy spectrum is not made of delta peaks, except in degenerate cases, where either E or $\tau^2 a$ is an integer multiple of π.

Our numerical investigations have been mainly concerned with the polarization of the system. If

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

denotes the state vector, this quantity is defined by

$$\mathbf{P} = \langle \psi | \boldsymbol{\sigma} | \psi \rangle \quad \left| \begin{array}{l} P_x = \psi_+^* \psi_- + \psi_-^* \psi_+ \\ P_y = i(\psi_-^* \psi_+ - \psi_+^* \psi_-) \\ P_z = \psi_+^* \psi_+ - \psi_-^* \psi_- \end{array} \right. \tag{26}$$

These three components are not independent, since the condition $\mathbf{P}^2 = 1$ is preserved by the dynamics for any spin-1/2 system. Hence, the polarization is advantageously described by two polar angles, according to

$$\left| \begin{array}{l} P_x = \sin\theta \cos\varphi \\ P_y = \sin\theta \sin\varphi \\ P_z = \cos\theta \end{array} \right. \tag{27}$$

For the "kicking" perturbation defined above, the polarization jumps in a discontinuous way at integer times and remains constant between two consecutive integer times. Let $\mathbf{P}(n)$ denote its value for $n < t < n+1$. Our three numerical approaches (Poincaré sections, polarization fluctuation, and nature of the quasi-energy spectrum) have been concerned with this sequence $\mathbf{P}(n)$. The initial condition $\psi_{t=0} = \binom{1}{0}$ is assumed throughout the following.

4. POINCARE SECTIONS

Consider the integrated amplitude of the perturbation up to time T

$$S_T = \sum_{n=1}^{T} a_{\nu(n)} \tag{28}$$

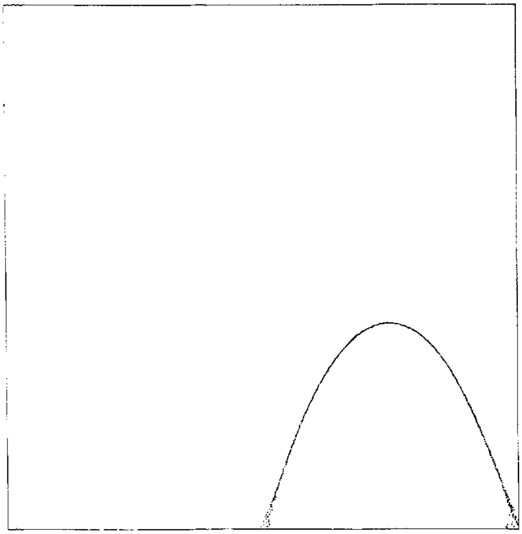
If the sequence $a_{\nu(n)}$ had zero average and was periodic, with period λ, S_T would vanish when T is a multiple of λ. Hence, it seems reasonable to define the quasiperiods of the present perturbation as the times T such that $|S_T| < a\varepsilon$, ε being some very small number. Figure 1 presents some Poincaré sections obtained by plotting the values of the polar angles θ and φ for the times T defined above. The value $\varepsilon = 10^{-3}$ selects 784 times such that $T \leqslant 10^6$. The plots correspond to different values of E at a fixed generic perturbation strength $a = 1$. Since the evolution is not quasiperiodic, it is expected that these plots are never curves, but rather that the data present some intrinsic scatter. This is indeed what can be seen. For small E, there is a very long crossover time before the scattering shows up. The first plot ($E = 10^{-3}$) indeed hardly looks different from a curve at the time scale $t = 10^6$. It can be argued, in analogy with what has been shown in the context of harmonic excitations in quasicrystals,[16] that the crossover time blows up exponentially as E, or a, goes to zero.

5. POLARIZATION FLUCTUATION

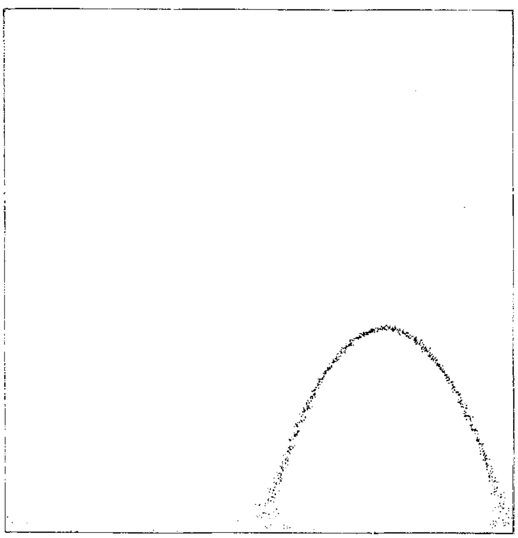
Let us restrict the discussion to systems in which none of the values $\sigma_z = +1$ or $\sigma_z = -1$ is preferred. This is indeed the case in the present example. We then define the polarization fluctuation by the formula

$$\Sigma_T = \sum_{n=1}^{T} P_z(n) \tag{29}$$

Since the average polarization Σ_T/T vanishes for $T \to \infty$, Σ_T grows less rapidly than T. It is clear that the fluctuation will be bounded in the case of a periodic evolution. This will remain true for a smooth enough quasiperiodic evolution. Conversely, with a random driving force, the fluctuation is expected to increase typically as $T^{1/2}$. In ref. 17 it has been shown

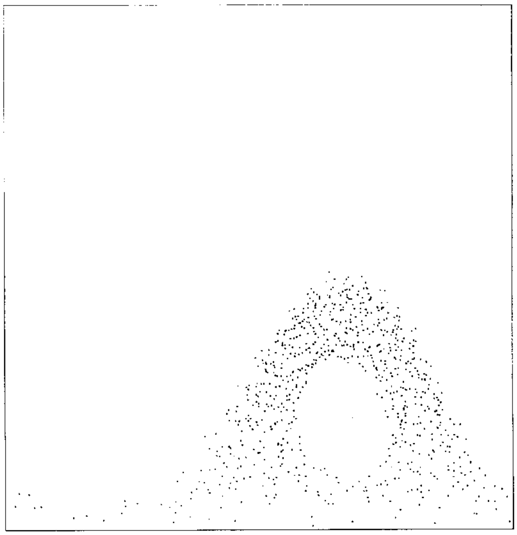


(a)

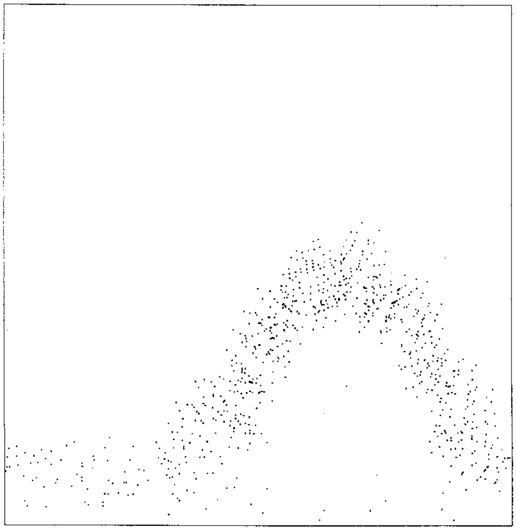


(b)

Fig. 1. Plots of Poincaré sections, defined in the text, in the plane of the polar angles, for $a = 1$ and different values of E: (a) $E = 0.001$, (b) $E = 0.01$, (c) $E = 0.05$, and (d) $E = 0.1$. For each plot, φ varies from $-\pi$ to π along the abscissa axis and θ from 0 to π along the ordinate axis.



(c)



(d)

Fig. 1 (*continued*)

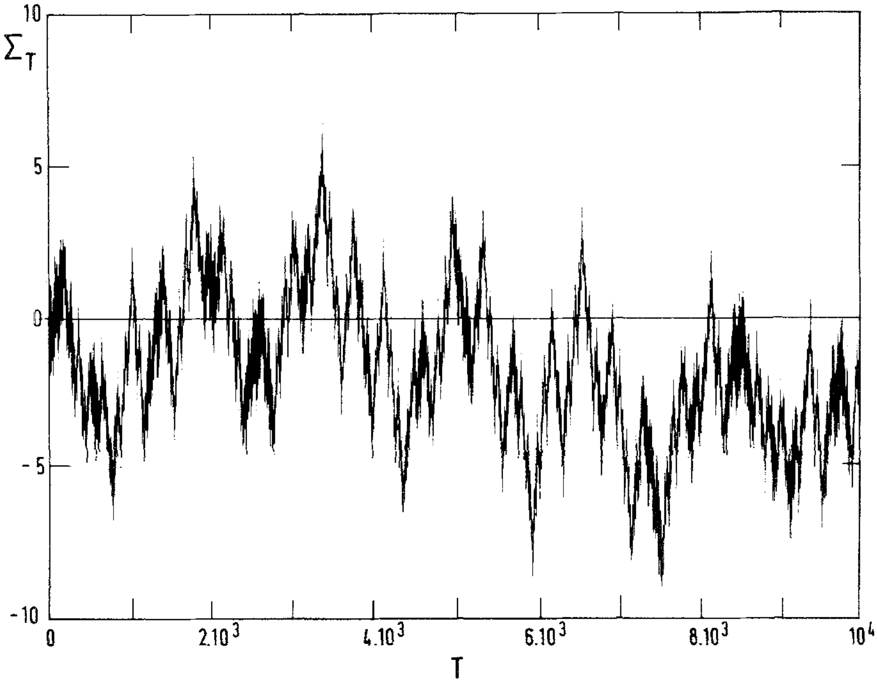


Fig. 2. Plot of the polarization fluctuation, defined in Eq. (29), up to $T = 10^4$, for $E = \pi/3$ and $a = 3\pi/5$.

that the density fluctuation, analogous to Σ_T, of the atomic positions of a quasiperiodic chain of atoms generated by a circle map typically exhibits a logarithmic divergence. This rather subtle effect originates in number theory. In the present case, such an exact treatment is not possible. We have computed the fluctuation Σ_T numerically for several values of E and a. Although the data are far from having the clear-cut scaling behavior of ref. 17, they are compatible with a divergence of a logarithmic type in the polarization fluctuation. Figure 2 shows a plot of Σ_T up to $T = 10^4$, for typical values of the parameters. The maximal and minimal fluctuations are still of the order of a few units on that time scale.

6. NATURE OF THE QUASI-ENERGY SPECTRUM

It has been shown that the quasi-energy spectrum (Fourier transform of the evolution operator) is not made up of delta peaks for generic values of the parameters. In other words, this spectrum is continuous. We now address the question of whether it is absolutely continuous (with a smooth density) or singular continuous. We use a numerical method introduced in ref. 18. In order to avoid dealing with matrix quantities, we restrict ourselves to the Fourier transform of the polarization P_z. We introduce the Fourier amplitude

$$G_T(\omega) = \sum_{n=1}^{T} e^{in\omega} P_z(n) \tag{30}$$

and the associated intensity $\mathscr{I}_T(\omega) = (1/T)\,|G_T(\omega)|^2$. The key observation is that this last quantity converges, as $T \to \infty$, to a function $\mathscr{I}(\omega)$ in the case of an absolutely continuous spectrum, and to a more singular object (generalized function, distribution) if the spectrum is singular continuous. Hence, the integrals

$$I_T = \int_0^{2\pi} \frac{d\omega}{2\pi} \mathscr{I}_T^{1/2}(\omega) \tag{31}$$

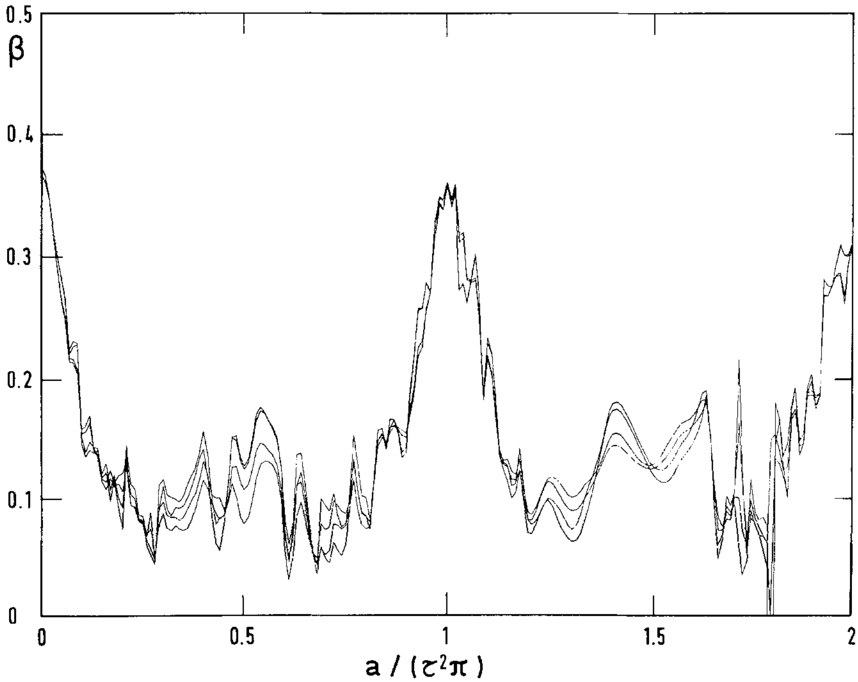


Fig. 3. Plot of the exponent β of the scaling law (32), extracted from data up to $T = 1600$, versus $a/\tau^2\pi$, for $E = \pi/3$. The scatter between the different curves gives a hint on the accuracy of the law (32) in the range of values of T that we have explored. The steep maxima at integer values of the abscissa correspond to degenerate models, where the value $\beta = 1/2$, up to a logarithmic correction, is expected.

have a nonvanishing $T \to \infty$ limit if, and only if, the spectrum has an absolutely continuous part. Conversely, it can be shown that $I_T \sim T^{-1/2}$ for a periodic, or smooth quasiperiodic, evolution. An intermediate kind of behavior, such as

$$I_T \sim T^{-\beta} \qquad \text{with} \quad 0 < \beta < 1/2 \tag{32}$$

can be expected in the case of a singular continuous spectrum, and has indeed been observed in ref. 18, in the case of the spatial Fourier transform of some one-dimensional atomic structure. We have computed the integrals I_T for numerous values of the perturbation strength a at fixed $E = \pi/3$, and extracted from the data in the range $100 \leqslant T \leqslant 1600$ an effective exponent β according to Eq. (32). Figure 3 shows these approximate values of β plotted versus $a/\tau^2\pi$. The different curves, obtained by using slightly different fitting procedures and discarding or not the data corresponding to the smaller values of T, coincide in a satisfactory way, suggesting the validity of the power law (32). The integer values of the abscissa $a/\tau^2\pi$ correspond to steep maxima for the exponent β. Indeed, the invariant I of Eq. (23b), given by Eq. (25) in the present case, vanishes for such values of a, which correspond to a quasiperiodic evolution, and yield $\beta = 1/2$.

7. CONCLUSION

To summarize, we have shown analytically that a two-level quantum system driven by a Fibonacci-like quasiperiodic perturbation generically does not have a quasiperiodic time evolution. Three different numerical investigations suggest an intermediate kind of evolution between quasiperiodic and chaotic, characterized by fuzzy Poincaré sections, a slowly growing polarization fluctuation, and a singular continuous quasi-energy spectrum.

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